Numerical Evaluation of Hierarchical Vector Finite Elements on Curvilinear Domains in 2-D

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***Abstract*—When using the finite element method, computational efficiency may be improved by using higher order elements. Typical 2-D triangular rectilinear elements cannot exactly model curved geometries, negating the advantage of higher order elements when such geometries are modeled. Curvilinear elements can be applied to such problems, and have significantly better computational efficiency than rectilinear elements when higher order elements are used. Results are shown for elemental basis orders ranging from CT/LN through full fifth order in rectilinear and curvilinear domains.**

***Index Terms*—Basis functions, curvilinear elements, finite element methods (FEMs).**

I. Introduction

THE finite element method (FEM) has seen significant use in electromagnetics to date [1]–[3]. It is well known that using higher order elemental basis functions result in improved solution efficiency, and that the asymptotic error behavior of the functional is $O(h^{2(q+1)})$, where the elemental basis is curl-complete to polynomial order q, h is the maximum edge length used to discretise the problem, and the standard curl-curl form of the vector wave equation is used to formulate a solution [4]. Since eigenvalues are stationary [5], their errors will converge at the same rate. The expected $O(h^{2(q+1)})$ holds when the geometric shape of the elements used conform exactly to the problem geometry. When geometry is not exactly modeled, an additional geometrical approximation error term could affect the asymptotic error behavior.

Triangular and tetrahedral elements are commonly used to discretise resp. 2-D and 3-D problems, and have fairly flexible geometrical modeling capabilities. Triangles can conform exactly to any polygonal geometry, and similarly tetrahedrons to any polyhedral geometry. However, when geometries with curved (i.e. nonrectilinear) shapes are modeled, a geometrical approximation must necessarily be made. As the order, p, of the elemental basis used to model the field is increased, the error contribution made by the geometrical modeling increases, eventually dominating the total error [6]. It is possible to define elements with curved sides [2, Sec. 7], typically using polynomial geometrical shape functions (e.g., [7]) to approximate the boundary of a problem. Using appropriate geometrical shape functions, it is in fact possible to represent many common geometries exactly [8].

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In this paper the performance of elements with progressively higher order bases in curved domains are evaluated in the context of eigen analysis of axially invariant homogeneous waveguides. Performance is evaluated by comparing the numerical performance with known analytical solutions over several modes. The solution efficiency on a rectilinear domain is also considered, providing a basis of comparison for the efficiency on curvilinear domains.

The effects of basis order, full vs. reduced-gradient basis spaces and in the case of guides with curved geometries, the effectiveness of various geometric mappings are considered. The main contribution in this paper is results for very high order elements in conjunction with curvilinear elemental geometry modeling.

II. FEM Formulation

The waveguide problem is formulated in terms of the curl-curl form of the vector Helmholtz operator [1]. Lossless homogeneous waveguides of constant $x-y$ cross-section with an assumed $e^{-jk_z z}$ field dependence in z are considered, resulting in a 2-D FEM domain.

The domain was discretised using Webb's hierarchical elements [4] up to full fifth order. The integrals were evaluated using Gaussian quadrature [9], [10] of order sufficient to exactly integrate the chosen basis. The resultant generalized eigenproblem was solved using LAPACK [11].

III. Curvilinear Elements

In order to use curvilinear elements, a tractable way of defining basis functions over them is needed. This can be done by introducing a curvilinear coordinate system [12]. Using curvilinear coordinates, vector values may be expressed in terms of *covariant* or *contravariant* components. Curl-conformance may be enforced by using the covariant components [13].

Evaluation of the elemental matrices require a change of integration variables to reference coordinates, and the evaluation of the curl operator in reference coordinates. This is shown in [14]. Suitable curvilinear mappings also have to be found.

A quadratic Legendre mapping, as well as two parametric curve mappings, respectively a cubic polynomial and a quadratic rational mapping, are considered. The elemental boundaries are mapped to parametric curves using Zlámal's mapping [15]. The quadratic rational mapping is able to exactly model any conic section [16], hence the geometrical modeling is numerically exact for the geometries considered in this paper. Detailed descriptions of these mappings are available in [17].

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TABLE I
BASIS NAME CONTRACTIONS USED

Contraction	Tangential Order	Normal Order
CT/LN	Constant	Linear
LT/LN	Linear	Linear
LT/QN	Linear	Quadratic
QT/CuN	Quadratic	Cubic
CuT/CuN	Cubic	Cubic
CuT/Q4N	Cubic	Quadric
Q4T/Q5N	Quadric	Quintic
Q5N/Q5N	Quintic	Quintic

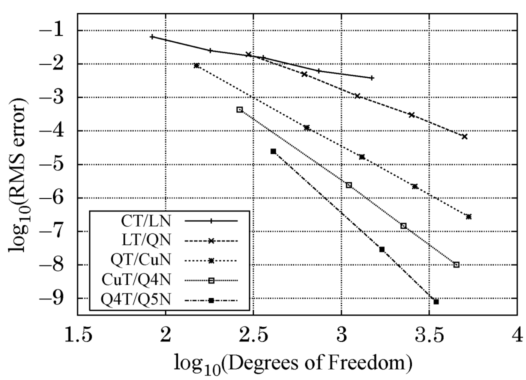

Fig. 1. RMS error over 12 TE modes of unmapped elements on a 1 × 0.5 m rectangular waveguide.

IV. CONVERGENCE STUDY IN RECTILINEAR DOMAIN

Convergence of the unmapped elements (i.e. triangular) elements is evaluated using a 1 by 0.5 m rectangular waveguide. Analytical solutions of this geometry are readily available (e.g., [18]). In this case there is no geometrical approximation, and the basis functions are undistorted. As such this represents a best case in terms of expected solution accuracy. Accuracy is evaluated by calculating the RMS error of the computed results over the first 12 respective TE and TM modes. The abbreviations used for basis orders are shown in Table I.

The RMS error of the computed results using reduced-gradient space basis functions are shown in Fig. 1. The x axis shows the number of unknowns, n. The triangle edge length, h, is approximately inversely proportional to $\sqrt{n}$, on a 2D domain. The convergence rate with respect to h is thus twice as fast as with respect to n.

All the element basis orders show the respective $O(h^{2(q+1)})$ convergences expected. The higher order elements are clearly more efficient per degree of freedom. Efficiency per degree of freedom is determined only by the basis order [19]. Total solution efficiency depends heavily on matrix solution speed, which in turn depends on the condition of the system matrices. Higher order bases tend to result in more poorly conditioned matrices [20], [21], which makes the advantage of the higher order functions less clear cut. However, using the appropriate definitions for degrees of freedom and a suitable preconditioned matrix solver, higher order elements are found to be significantly more efficient [19], [22].

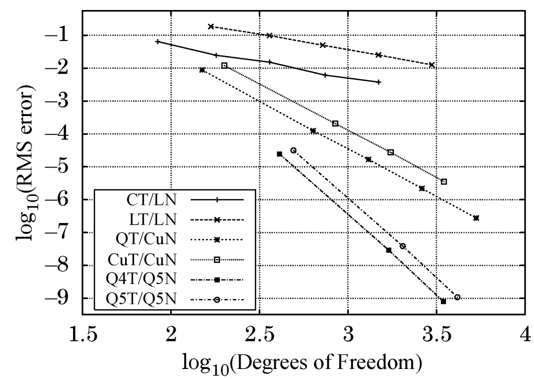

Fig. 2. RMS error over 12 TE modes of unmapped elements with full and gradient reduced bases on a 1 × 0.5 m rectangular waveguide.

The TM modes, not shown here, exhibited similar error trends. However, the TM errors are consistently larger than TE errors corresponding to the same basis order and mesh. The fact that the TM modes have higher cutoff frequencies and the weak enforcement of the H wall boundary conditions probably account for this difference.

While full-order and gradient reduced basis spaces should both converge at the same rate when a curl-curl formulation is used, whether the one or the other is optimal is problem dependant [4]. For homogeneous waveguides it seems natural that reduced gradient bases should be more efficient, and in Fig. 2 this can be seen to be the case. While each full-order base converges at the same rate as its reduced gradient counterpart as expected, the full-order bases mostly show worse errors than their gradient reduced counterparts when the exact same mesh is used. A possible explanation is advanced in [23]. Only the TE modes are shown since, again, the TM results are similar.

V. RESULTS IN CURVILINEAR DOMAINS

Circular and elliptic waveguides were considered as examples of curvilinear domains. Analytical results for circular waveguides are readily obtainable, e.g., [18], while the analytical solution of elliptic guides is described in [24]. A circular waveguide with a radius of 1 m was considered, and an elliptic guide with a 1 m main axis, and an ellipticity of 0.5. The TE and TM modes of the circular waveguide are both considered, while only the TE modes of the elliptical waveguide are considered.

The results for a circular waveguide using unmapped (i.e. rectilinear triangular) elements are shown in Fig. 3. Elements of all order converge at the same rate as the CT/LN elements, while all the higher order elements are less efficient than the CT/LN elements. It is clear that the solution accuracy of higher order elements is limited by the geometrical error, and that the geometrical error converges at about the same rate as the field error of CT/LN elements.

Lower order elements require more mesh elements to generate the same number of unknowns as higher order elements, resulting in the lower order elements having better geometric modeling for a given number of unknowns, which results in the CT/LN elements being the most accurate in this case.

The coarsest mesh consist of six identical triangles, except for the CT/LN basis. A six element CT/LN mesh does not have sufficient degrees of freedom to generate all 12 eigen-modes used

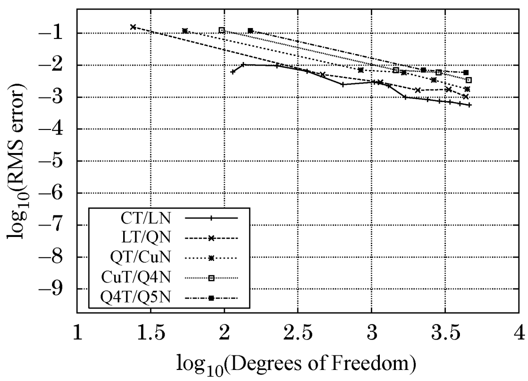

Fig. 3. RMS error over 12 TE modes of unmapped elements on a 1 m radius circular waveguide.

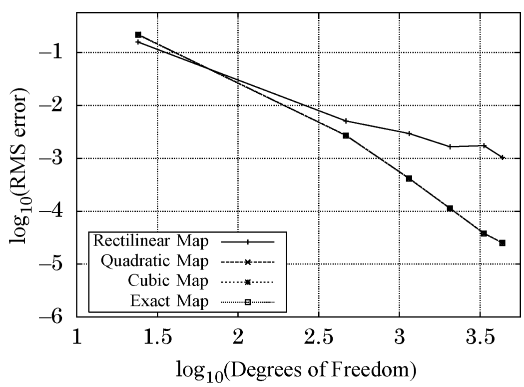

Fig. 4. RMS error over 12 TE modes of the LT/QN basis on a 1 m radius circular waveguide. The three curvilinear results lie on top of each other.

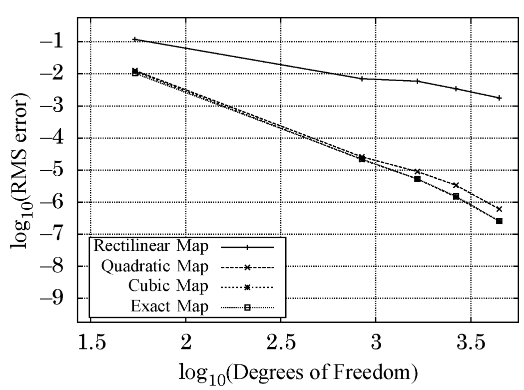

Fig. 5. RMS error over 12 TE modes of the QT/CuN basis on a 1 m radius circular waveguide.

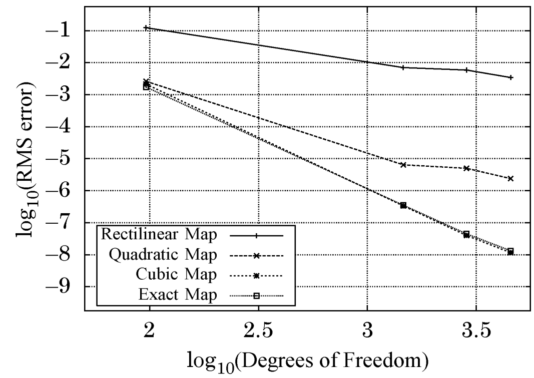

Fig. 6. RMS error over 12 TE modes of the CuT/Q4N basis on a 1 m radius circular waveguide.

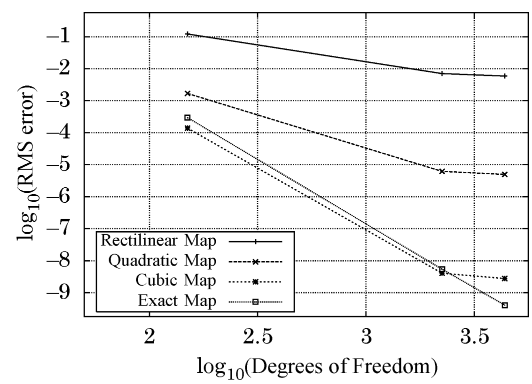

Fig. 7. RMS error over 12 TE modes of the Q4T/Q5N basis on a 1 m radius circular waveguide.

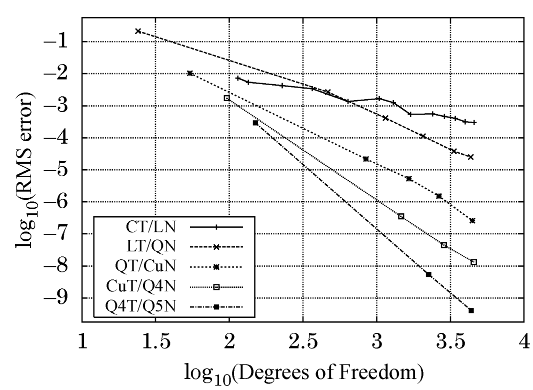

Fig. 8. RMS error over 12 TE modes using the quadratic rational mapping on a 1 m radius circular waveguide.

for comparison. All meshes used consisted of nearly homogeneous triangular elements.

The mappings are compared in Fig. 4through Fig. 7, using mixed order bases of order LT/QN through Q4T/Q5N. With the LT/QN basis, Fig. 4 shows all the curvilinear mappings performing identically. The curvilinear mappings cause a slightly greater error when the coarsest mesh is considered. With the QT/CuN basis, Fig. 5 shows all the curvilinear mappings performing similarly, with the quadratic map being slightly worse. Increasing the basis order to CuT/Q4N, the exact and cubic maps are seen to perform equally well in Fig. 6, while the quadratic map is clearly inferior. In Fig. 7 the cubic mapping is shown to inhibit the convergence rate of the Q4T/Q5N base, though conversely, it performs better than the exact mapping on the coarsest (6 element) mesh.

In both Fig. 4 and Fig. 7 it is seen that the most accurate geometrical map does not necessarily lead to the most accurate result under all circumstances. This may be because the geometrically more accurate maps have different basis function distortion characteristics [6].

Results for full-order vs. mixed order, and TM modes were generated, but are not shown. The comments made in Section IV apply directly to the curvilinear results, provided that a mapping of sufficient order is used.

Fig. 8 shows the convergence of all the mixed order basis functions considered using the exact mapping. This is the equivalent of Fig. 1 for the curved domain. It can be seen that using

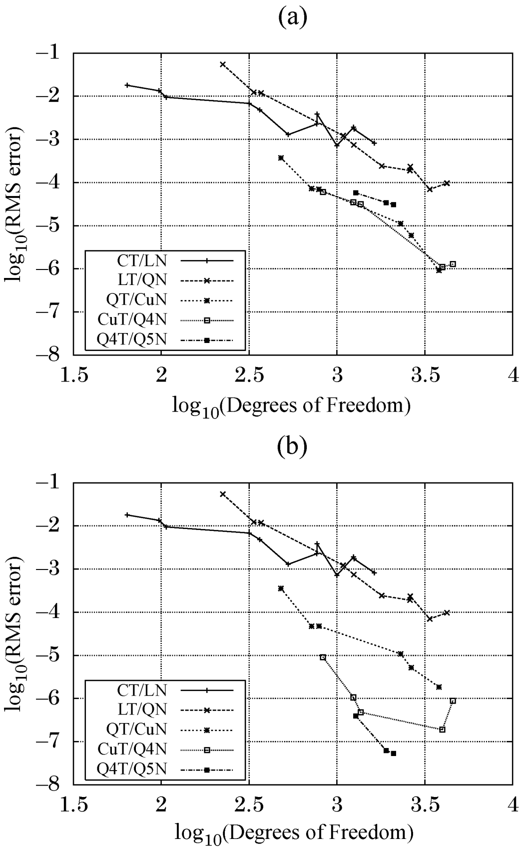

Fig. 9. Performance of curvilinear mappings on an elliptic waveguide, with major axis 1 m, ellipticity 0.5, RMS error over 12 TE modes. (a) Quadratic mapping; (b) cubic mapping.

the appropriate mapping function, the convergence properties of the bases are maintained even on curved domains.

In order to verify the validity of the results on a different geometry, an elliptic waveguide was also considered. Results for the elliptic guide are shown in Fig. 9. Only the TE modes and the quadratic and cubic mappings are considered. The general trends are very similar to what is seen with the circular guide. Again the quadratic mapping constrains the three highest order element types, while the cubic mapping significantly improves the relative performance of the two highest order element types. The graphs are less monotonic than for the circular waveguides, since the mesher was unable to produce highly homogeneous meshes.

VI. Conclusion

It was seen that higher order elements provide significant improvement in computational efficiency as compared to the standard CT/LN Whitney element. However, when the elemental shape can not conform to the geometry of the problem, the higher order elements provide no advantage. It was seen that using curvilinear elements dramatically improve the effectiveness of higher order elements when curved problem domains are considered. It was also seen that the geometrical approximation chosen needs to be of sufficient order and precision to prevent it from constraining the final accuracy, though other factors come into play in pre-asymptotic solutions.

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